REV-01
MSM/07/12

2023/06

SET A

M.Sc. MATHEMATICS
FOURTH SEMESTER
ADVANCE ALGEBRA
MSM - 404C
[USE OMR FOR OBJECTIVE PART]

Duration: 3 hrs. **Full Marks: 70**

(Objective)

Time: 30 min. **Marks: 20**

Choose the correct answer from the following: ***1X20=20***

1. Let $B = \{\{1\},\{3\},\{1,2\},\{2,3\}\}$. Then $(B, \subseteq)$ is a poset having
   a. greatest element but not the least element
   b. least element but not the greatest element
   c. neither the greatest element nor the least element
   d. both the greatest element nor the least element

2. Consider $P = \{2,3,4,6\}$ with partial order $\leq$ defined by
$$a \leq b \text{ if and only if } a \text{ divides } b$$
then
   a. 2 is a minimal element of P but 3 is not
   b. 3 is a minimal element of P but 2 is not.
   c. Both 2 and 3 are minimal elements of P
   d. None of the above statements is true

3. If L be any lattice and $a, b \in L$ then
   a. $a \wedge b \geq a$
   b. $a \wedge b \geq b$
   c. $a \vee b \geq a$
   d. $a \vee b \leq a$

4. If P be a property of a free group. Then P is true if P denotes
   a. No free group is torsion free
   b. A free group on $X = \{x, y\}$ is abelian.
   c. A free group on $X = \{x\}$ is finite cyclic.
   d. A free group on $X = \{x\}$ is infinite cyclic.

5. In any lattice L and for any $a, b, c \in L$ if $a \geq b$ then
   a. $a \wedge (b \vee c) \leq b \vee (a \wedge c)$
   b. $a \wedge (b \vee c) \geq b \vee (a \wedge c)$
   c. $a \wedge (b \vee c) < b \vee (a \wedge c)$
   d. $a \wedge (b \vee c) > b \vee (a \wedge c)$

6. A poset P is a toset (totally ordered set) if
   a. Only two elements in P are comparable.
   b. Every two elements in P are comparable.
   c. No two elements in P are comparable.
   d. None of these

7. The statement that '$\mathbb{Z} \times \mathbb{Z}$ is a free abelian group' is
   a. False
   b. True
   c. Doubtful
   d. None of these

8. If $\mathbb{Z}$ be the additive abelian group of integers then a basis for $\mathbb{Z} \times \mathbb{Z}$ is
   a. $\{(4,1),(1,0)\}$
   b. $\{(3,0),(1,0)\}$
   c. $\{(0,3),(0,1)\}$
   d. $\{(0,4),(0,2)\}$

USTM/COE/R-01

9. If $X = \{x, y\}$ then w is a reduced word in $X \cup X^{-1}$ where

a. $w = xxy^{-1}y^{-1}xx^{-1}x$

b. $w = xyxyxy^{-1}yx$

c. $w = xyyxy^{-1}yxy$

d. $w = xyx^{-1}yx^{-1}$

10. An abelian group G is said to be free abelian group if there is a set $X \subset G$ such that

a. X is linearly independent

b. X generates G

c. X is both linearly independent and a generating set for G

d. None of these

11. Let $X = \{x, y\}$. The word x^{-1} is adjacent to

a. $x^{-1}xyy^{-1}y$

b. $x^{-1}xx^{-1}$

c. $yyyxy^{-1}$

d. 1

12. Let $X = \{x, y\}$. The set of all words in X forms

a. a group

b. an Abelian group

c. a semi group

d. None of these

13. If M is an irreducible unital R-module then for any $x \in M$

a. $Rx \subset M$

b. $M \subset Rx$

c. $Rx = M$

d. None of these

14. Let R be a subring of a ring S. Then

a. R is a left (right) S-module

b. S is a left (right) R-module

c. Zero subring (0) is not a left (right) S-module

d. None of these

15. A lattice L is said to be complete if every non empty subset of L has

a. greatest element

b. least element

c. both greatest and least elements

d. both supremum and infimum in L

16. For any ring R a left R-module M is Noetherian if

a. M has acc (ascending chain condition)

b. M has dcc (descending chain condition)

c. M has both acc and dcc

d. None of these

17. The submodules of a quotient module M/N are of the form U/N where U is a submodule of

a. M

b. N

c. M containing N

d. N containing M

18. If M is an R-module then the set $RM = \{\sum r_i m_i \ : \ r_i \in R,\ m_i \in M\}$ is

a. an M-module

b. a submodule of M

c. a submodule of R

d. None of these

19. For a module M if every non-empty set S of submodules of M has a maximal element, then

a. M is Artinian

b. M is Noetherian

c. M is both Artinian and Noetherian

d. None of these

20. Let $G \neq \{0\}$ be a free abelian group with a finite basis. If B and B' are two bases for G, then

a. both B and B' are finite and $|B| > |B'|$

b. both B and B' are finite and $|B| < |B'|$

c. both B and B' are finite and $|B| = |B'|$

d. both B and B' are not necessarily finite.

(Descriptive)

Time : 2 hrs. 30 mins. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. Define each of the following terms with illustration in the context of a poset: 4+2+4 =10
   (i) Greatest element
   (ii) Least element
   (iii) Maximal element
   (iv) Minimal element

   b. When is a poset called lattice?

   c. Consider the poset $(X, \leq)$ where $X = \{1, 2, 3, 4, 6, 12\}$ and for $a, b \in X, a \leq b$ if a divides b. Examine if $(X, \leq)$ is a lattice or not.

2. a. Establish the distributive inequality 4+1+5 =10
   $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
   in any lattice $(L, \leq)$, where $a, b, c \in L$.

   b. When is a lattice $(L, \leq)$ called distributive?

   c. Show that $(P(X), \subseteq)$ is a distributive lattice, where $P(X)$ is a power set of any non empty set X.

3. a. Let the group G have subgroups G_i for $1 \leq i \leq n$ such that 6+4=10
   (i) G_i commutes with G_j for $1 \leq i < j \leq n$.
   (ii) $G = G_1 G_2 \cdots G_n$
   (iii) $G_i \cap G_1 G_2 \cdots G_{i-1} G_{i+1} \cdots G_n = \{e\}$, e being the identity of G.
   Then prove that $G \cong G_1 \times G_2 \times G_3 \times \cdots \times G_n$.

   b. Explain the concepts of cartesian product and direct product of an arbitrary family of groups $\{G_\lambda : \lambda \in \Lambda\}$, where Λ is an arbitrary indexed set.

4. a. What is a free abelian group? If G is a non zero free abelian group with a basis of r elements, then show that G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for r-factors. 2+4+4 =10

   b. Examine $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}_n$ are free abelian or not.

5. a. When are two free groups defined respectively on two sets X and Y be isomorphic? Show that every free group is torsion free. 1+4+5 =10

b. Let F_m and F_n be free groups of finite ranks m and n respectively. Then prove that $F_m \cong F_n$ if and only if $m = n$.

6. a. Explain the concepts of a left module and a right module over a ring R with unity. When will a left module be also a right module? Give two examples of R-module M where R is a ring. 2+1+2+5=10

b. If $(N_i)_{i\in\Lambda}$ is a family of R-submodules of a module M then show that

$$\sum_{i\in\Lambda} N_i = \{x_{i1} + x_{i2} + \cdots + x_{ik} \ : \ x_{ij} \in N_{ij}\,, ij \in\Lambda\}$$

7. Define the direct sum of a family $(N_i)_{i\in\Lambda}$ of R-submodules. Let $(N_i)_{i\in\Lambda}$ be a family of a R-submodules of an R-module M. Prove that the following statements are equivalent: 3+7=10

(i) $\sum_{i\in\Lambda} N_i$ is a direct sum

(ii) If $x_{i1} + x_{i2} + \cdots + x_{ik} = 0, x_{ij} \in N, ij \in\Lambda$ then $x_{i1} = 0, x_{i2} = 0, \cdots, x_{ik} = 0$.

(iii) $N_j \cap \sum_{i\neq j} N_i = 0, \forall j \in\Lambda$

8. a. When is a left R-module M called notherian? Is the ring $\mathbb{Z}$ of all integers notherian? Give justification ($\mathbb{Z}$ may be considered as a module over $\mathbb{Z}$). 2+2+6 =10

b. If every submodule of a module M is finitely generated then prove that every non-empty collection C of submodules of M has a maximal element.

= = *** = =