SET B

M.Sc. MATHEMATICS
THIRD SEMESTER
FUNCTIONAL ANALYSIS
MSM – 302

[USE OMR SHEET FOR OBJECTIVE PART]

Duration : 3 hrs. **Full Marks : 70**

(Objective)

Time: 30 min. **Marks: 20**

Choose the correct answer from the following: ***1X20=20***

1. In any normed space $(V, \|\cdot\|)$ with $u, v \in V$
   a. $|\| u \| - \| v \|| < \| u - v \|$
   b. $|\| u \| - \| v \|| > \| u - v \|$
   c. $|\| u \| - \| v \|| \geq \| u - v \|$
   d. $|\| u \| - \| v \|| \leq \| u - v \|$

2. The true statement of the following is
   a. There are norms on a finite dimensional norm space which are not equivalent to each other.
   b. Convergence or divergence of a sequence in a finite dimensional normed space depends on particular norm defined on the space.
   c. There is a finite dimensional normed space which is not a Banach space.
   d. A finite dimensional subspace of a normed space is closed.

3. Consider $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}, i = 1,2, \cdots, n\}$. For any $x = (x_1, x_2, \cdots, x_n)$ define $\|\cdot\|_p$ by $\| x \|_p = \{\sum_{i=1}^{n} |x_i|^p\}^{\frac{1}{p}}$. Then $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ if
   a. $1 < p < \infty$
   b. $1 \leq p < \infty$
   c. $0 \leq p < \infty$
   d. $-\infty < p < \infty$

4. The closure $\bar{Y}$ of a linear subspace Y of a normed space $(V, \|\cdot\|)$ is
   a. Linearly closed
   b. Linearly not a closed subspace
   c. Neither (a) nor (b) is true
   d. None of these

5. A proper subspace of a normed space has
   a. No any interior point
   b. At least one interior point
   c. At most one interior point
   d. Many interior point

6. In any inner product space X over a complex field, for any $x, y \in X$
   a. $< x, y > = < y, x >$
   b. $< x, y > = \overline{< x, y >}$
   c. $< x, y > = \overline{< y, x >}$
   d. None of these

7. For any non-empty subset A of an inner product space X
   a. $A^{\perp} \subset \overline{A^{\perp}}$
   b. $A^{\perp} \supset \overline{A^{\perp}}$
   c. $A^{\perp} = \overline{A^{\perp}}$
   d. None of these

8. In an inner product space X over a complex field
   a. $< \alpha x, y > = \bar{\alpha} < x, y >$
   b. $< x, \alpha y > = \alpha < x, y >$
   c. $< x, \alpha y > = \bar{\alpha} < x, y >$
   d. None of these

9. Let $1 \leq p < q < \infty$ and $l^p \& l^q$ have usual meaning. Then
   a. $l^p \subset l^q$
   b. $l^p \supset l^q$
   c. Both (a) and (b) are true
   d. Both (a) and (b) are false

10. Any two n- dimensional normed spaces are:

a. Algebraically non isomorphic

b. Topologically non-isomorphic

c. Topologically isomorphic

d. None of the these

11. Consider $l^p = \{z = \{z_n\}_{n=1}^{\infty},\ z_n \in \mathbb{C}\}$ for $1 \le p < \infty$. Define $\| z \|_p = \{\sum_{n=1}^{\infty} |z_n|^p\}^{1/p}$. Then $\|\cdot\|_p$ will be a norm on l^p if

a. $\sum_{n=1}^{\infty} |z_n|^p < \infty$

b. $\sum_{n=1}^{\infty} |z_n|^p < 0$

c. $\sum_{n=1}^{\infty} |z_n|^p \le 0$

d. None of these

12. If $A \neq \phi$ is a subset of an inner product space X, then

a. $A^{\perp} \subset A^{\perp\perp\perp}$

b. $A^{\perp} = A^{\perp\perp\perp}$

c. $A^{\perp} \supset A^{\perp\perp\perp}$

d. None of these

13. Which of the following statements is false:

a. Linear operator on a finite dimensional normed space is continuous.

b. Linear operator on a finite dimensional normed space is bounded.

c. Linear operator on a finite dimensional normed space is both continuous and bounded.

d. None of the these

14. If B and B' are Banach spaces and $T: B \to B'$ is a continuous linear operator then

a. T is an open mapping

b. T is not an open mapping

c. Both (a) and (b) are doubtful

d. None of the above is true

15. Two normed spaces X and Y are said to be topologically isomorphic if there is a mapping $T: X \to Y$ such that

a. T is linear and T is not a homeomorphism

b. T is a homeomorphism and T is not linear

c. T is both linear and a homeomorphism

d. T is neither linear nor a homeomorphism

16. Which of the following statements is false?

a. If c denotes the Banach space of all convergent sequences in $\mathbb{R}$ or $\mathbb{C}$ and c_0 denotes the Banach space of all convergent sequences converging to 0, then c_0 is not a closed subspace of c.

b. If Y be a complete subspace of a normed space X, then Y is closed in X.

c. If Y be a complete subspace of a Banach space X then Y is also a Banach space.

d. All of the above statement are true.

17. Two normed spaces X and Y over the same field will be isometric if there is a linear operator $T: X \to Y$ such that

a. $\| T(x) \|_Y > \| x \|_X, \quad \forall x \in X$

b. $\| T(x) \|_Y < \| x \|_X, \quad \forall x \in X$

c. $\| T(x) \|_Y = \| x \|_X, \quad \forall x \in X$

d. $\| T(x) \|_Y \neq \| x \|_X, \quad \forall x \in X$

18. Every convergent sequence in a normed space is a Cauchy sequence, but every Cauchy sequence in it may not be a convergent sequence. The statement is

a. True
b. False
c. Not decidable
d. None of these

19. If T is a linear operator from a normed space X to a normed space Y over the same field K then which of the following statements is not true?

a. T is continuous if T is bounded
b. T is bounded if T is continuous
c. T is continuous if and only if T is bounded
d. None of the above is true.

20. If T be a bounded linear operator from a normed space X into a normed space Y over the same field K then the norm of T is given by

a. $\| T \| = \text{Sup}\{\| T(x) \|_Y : x \in X, \| x \|_X > 1\}$
b. $\| T \| = \text{Sup}\{\| T(x) \|_Y : x \in X, \| x \|_X \geq 1\}$
c. $\| T \| = \text{Sup}\{\| T(x) \|_Y : x \in X, \| x \|_X \leq 1\}$
d. None of these

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Descriptive

Time : 2 hrs. 30 mins. **Marks : 50**

[Answer question no.1 & any four (4) from the rest]

1. a. Define a normed linear space $(X, \|\cdot\|)$. **2+4+4 =10**
   b. Show that in a normed linear space $(X, \|\cdot\|)$,
   $|\| x \| - \| y \|| \leq \| x - y \|, \forall x, y \in X$.
   c. Using result (b) prove that $\|\cdot\| : X \to \mathbb{R}$ is a continuous function.

2. a. Consider $\mathbb{R}^n$ the linear space of all n-tuples of real numbers. For any $f \in \mathbb{R}^n, f = (f(1), f(2), \cdots, f(n))$, define $\| f \| = (\sum_{i=1}^{n} |f(n)|^2)^{\frac{1}{2}}$. Show that $\mathbb{R}^n$ is a normed space with $\|\cdot\|$ as defined. **5+5=10**
   b. Show that the normed space as defined in (a) is a Banach space.

3. a. Let Y be a subspace of a normed space X. If Y is complete then prove that it is also closed in X. **5+5=10**
   b. Let X be a normed space over a field K ($K = \mathbb{R}$ or $\mathbb{C}$) and let M be a closed subspace of X. Prove that quotient space X/M is a normed space under a suitably defined norm in X/M.

4. a. When is a linear operator $T: (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ said to be bounded. **2+4+4 =10**
   b. Prove that a linear operator $T: (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous if and only if T is bounded.
   c. Define the norm of a bounded operator $T: (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ and hence prove that the space of all bounded operators from $(X, \|\cdot\|_X)$ into $(Y, \|\cdot\|_Y)$ is also a normed space, the underlying field being $K(= \mathbb{R}$ or $\mathbb{C})$ for all normed spaces under consideration.

5. a. When are two normed spaces X and Y said to be topologically isomorphic. **2+8=10**

b. Let X and Y be normed spaces over the same field $K(=\mathbb{R}$ or $\mathbb{C})$ and let $T : X \to Y$ be an onto linear operator. Prove that T is a topological isomorphism if and only if there exist $K_1, K_2 > 0$ such that

$$K_1 \parallel x \parallel_X \leq \parallel T(x) \parallel_Y \leq K_2 \parallel x \parallel_X, \quad \forall x \in X$$

6. a. Any two n-dimensional normed spaces over the same field are topologically isomorphic. Justify the statement with a proof. **7+3=10**

b. Use (a) to show that all norms on a finite dimensional normed space are equivalent.

7. a. State Hahn Banach Theorem. **1+6+3 =10**

b. Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0 then show that f can be extended to a functional f_0 defined on M_0 such that $\parallel f_0 \parallel = \parallel f \parallel$.

c. Explain the concept of an inner product space.

8. a. If x and y are any two vectors in an inner product space X then establish the Schwarz inequality $| < x, \ y > | \leq \parallel x \parallel \parallel y \parallel$. **5+5=10**

b. Let M be a closed subspace of a Hilbert space H. Then show that $H = M \oplus M^{\perp}$, where $M^{\perp}$ is the orthogonal set of M.

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