REV-01
MSM/48/16/19

2022/12

SET C

M.Sc. MATHEMATICS
THIRD SEMESTER
NUMBER THEORY
MSM – 301
[USE OMR FOR OBJECTIVE PART]

Duration : 3 hrs. Full Marks : 70

(PART-A: Objective)

Time: 30 min. Marks: 20

Choose the correct answer from the following: ***1X20=20***

1. The congruence $x^2 \equiv 2 (\text{mod } p)$ has
   a. a solution for any odd prime p
   b. a solution for any odd prime $p \equiv 1(\text{mod } 8)$
   c. a solution for any odd prime $p \equiv 3(\text{mod } 8)$
   d. None ofthese
2. The value of x such that $3x \equiv 5 \ (\text{mod } 7)$ is
   a. 0
   b. 3
   c. 4
   d. 5
3. Which of the following primes satisfy the congruence $a^{24} - 6a - 2 \equiv 0(\text{mod } 13)$?
   a. 47
   b. 67
   c. 83
   d. None of these
4. Suppose ϕ and φ denotes Golden ratio and Golden ratio conjugate respectively. The nth Fibonacci number F_n is equal to
   a. $F_n = \dfrac{\phi^n - \varphi^n}{\sqrt{5}}$
   b. $F_n = \dfrac{\phi^n - (-\varphi)^n}{\sqrt{5}}$
   c. $F_n = \dfrac{(\phi)^n + (-\varphi)^n}{\sqrt{5}}$
   d. $F_n = \dfrac{(-\phi)^n + \varphi^n}{\sqrt{5}}$
5. Which of the following is/are true?
   a. 6 is the integer root of $x^2 + x + 1 \equiv 0 \ (\text{mod } 7)$
   b. 6 is not the integer root of $x^2 + x + 2 \equiv 0 \ (\text{mod } 7)$
   c. $6^7 \equiv 6(\text{mod } 7)$
   d. $6^7 \not\equiv 6(\text{mod } 7)$
6. If $C_k = \frac{p_k}{q_k}$ is the k-th convergent of $[a_0; a_1, a_2, \cdots, a_n]$ then which of the following is/are true?
   a. $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$, for $1 \le k \le n$
   b. $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k}$, for $1 \le k \le n$
   c. $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k+1}$, for $1 \le k \le n$
   d. None of these
7. The value of $[0; 1, 1, 1, 1, \cdots 1]$ is
   a. $\dfrac{F_{n+1}}{F_n}$
   b. $\dfrac{1-\sqrt{5}}{2}$
   c. $\dfrac{F_n}{F_{n+1}}$
   d. $\dfrac{1+\sqrt{5}}{2}$

8. The sequence $C_1, C_3, C_5, \cdots$ is

a. Decreasing sequence
b. Strictly decreasing sequence
c. Increasing sequence
d. Strictly increasing sequence

9. Which of the following numbers has no primitive roots?

a. 31
b. 125
c. 61
d. None of these

10. The unit digit of 2^{100} is

a. 2
b. 4
c. 6
d. 8

11. What is the smallest positive integer in the set $\{595x + 252y : x, y \in \mathbb{Z}\}$?

a. 1
b. 7
c. 252
d. 595

12. The congruence $x^6 \equiv 1 (\text{mod } 31)$ has

a. No solution
b. Exactly 6 solutions
c. At least 6 solutions
d. At most 6 solutions

13. Which of the following congruence has no solution

a. $39x \equiv 1 (\text{mod } 13)$
b. $6x \equiv 15\ (\text{mod } 21)$
c. $5x \equiv 2 (\text{mod } 26)$
d. None of these

14. Which of the following statement(s) is/are necessarily true?

a. $\phi(n) \mid n$ for all positive integers n.
b. $a \mid \phi(a^n - 1)$ for all positive integers a & n such that $\gcd(a, n) = 1$.
c. $n \mid \phi(a^n - 1)$ for all positive integers a & n such that $\gcd(a, n) = 1$.
d. None of these

15. For an odd prime p, the value of $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right)$ is

a. Always greater than 0
b. Equal to 0
c. An integer
d. None of these

16. If p_n denotes the nth prime number then which of the following is /are true?

a. $p_n \leq p_1 p_2 \cdots p_{n-1} - 1 \quad n \geq 2$
b. $p_n \leq p_1 p_2 \cdots p_{n-1} \quad n \geq 2$
c. $p_n \leq p_1 p_2 \cdots p_{n-1} + 1 \quad n \geq 2$
d. None of these

17. If $12x \equiv 4y\ (\text{mod } 30)$then which of the following is/are true?

a. $3x \equiv y\ (\text{mod } 30)$
b. $3x \equiv y\ (\text{mod } 5)$
c. $12x \equiv 4y\ (\text{mod } 5)$
d. None of these

18. The number of primitive roots of 17 is:

a. 16
b. 17
c. 8
d. Data is insufficient

19. The remainder obtained when 16^{2016} is divided by 9 equals

a. 1
b. 2
c. 3
d. 7

20. The remainder of $1! + 2! + 3! + \cdots + 100!$ upon dividing the sum by 12 is:

a. 0
b. 6
c. 9
d. None of these

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PART-B : Descriptive

Time : 2 hrs. 30 mins. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. Find all the positive integer solutions of $172x + 20y = 1000$. **4+3+3 =10**

   b. Find $\gcd(F_n, F_{n+1}), n \geq 1$ for the Fibonacci sequence F_n.

   c. Prove that the integer $111^{333} + 333^{111}$ is divisible by 7.

2. a. Find the common solution of the following system of linear congruence in the interval $[601, 800]$: **5+5=10**

$$x \equiv 2 \pmod{5}$$
$$x \equiv 3 \pmod{7}$$
$$x \equiv 4 \pmod{11}$$

   b. Solve the following congruence: $x^3 \equiv 4 \pmod{13}$.

3. Find the continued fraction expansion of **5+5=10**
   i. $\frac{118}{303}$
   ii. $\sqrt{23}$

4. Evaluate: **5+5=10**
   i. $[1; 1, 1, \cdots]$
   ii. $[1; 2, \overline{3, 1}]$

5. a. Solve the following linear congruence $140x \equiv 133 \pmod{301}$. **4+3+3 =10**

   b. Prove that- for any odd prime p,

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{\frac{(p+1)}{2}} \pmod{p}$$

   c. Verify that $4(29!) + 5!$ is divisible by 31.

6. a. Prove that - $\phi(2^n - 1)$ is a multiple of n for any $n > 1$. 3+4+3 =10

b. Verify that 3 is a primitive root of $F_n, n > 1$.

c. If p is an odd prime, then prove that

$$\left(\frac{-2}{p}\right) = \begin{cases} -1, & \text{if } p \equiv 5 \pmod 8 \text{ or } p \equiv 7 \pmod 8 \\ 1, & \text{if } p \equiv 1 \pmod 8 \text{ or } p \equiv 3 \pmod 8 \end{cases}$$

7. a. Find all the solution of $3x^2 + 9x + 7 \equiv 0 \pmod{13}$. 4+4+2 =10

b. Evaluate $\left(\frac{-42}{61}\right)$.

c. For an odd prime p, prove that the congruence $2x^2 + 1 \equiv 0 \pmod p$ has a solution if and only if $p \equiv 1$ or $3 \pmod 8$.

8. a. If p is a prime number and $d \mid (p - 1)$, then the congruence 5+2+3 =10

$$x^d - 1 \equiv 0 \pmod p$$

has exactly d solutions.

b. Prove that -The polynomial $f(n) = n^2 + n + 41$ is composite.

c. Prove That - If p_n is the nth prime number, then $p_n \leq 2^{2^{n-1}}$.

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