REV-01
MSM/07/13

2022/07

M.Sc. MATHEMATICS
FOURTH SEMESTER
ADVANCED ALGEBRA
MSM-404C

Duration : 3 hrs. **Full Marks: 70**

(PART-A: Objective)

Time : 20 min. **Marks : 20**

Choose the correct answer from the following: *1X20=20*

1. If L be any lattice and $a, b \in L$ then
   a. $a \wedge b \geq a$
   b. $a \wedge b \geq b$
   c. $a \wedge b \leq a$
   d. None of these

2. Let L be any lattice
   a. If L is distributive then it is modular
   b. If L is modular then it is distributive
   c. L is distributive if and only if it is modular
   d. None of these

3. Let P be a property of a free group. Then P is true if P denotes the property:
   a. Every free group is torsion group.
   b. A free group on $X = \{x, y\}$ is Abelian
   c. A free group on $X = \{x\}$ is finite cyclic
   d. A free group on $X = \{x\}$ is infinite cyclic

4. For any non empty set $X = \{x_\lambda : \lambda \in \Lambda\}$ every element of $X \cup X^{-1}$ is a word of length
   a. greater than 1
   b. greater than 2
   c. equal to 0
   d. equal to 1

5. A poset $(P, \leq)$ is a meet-semi lattice if for all $a, b \in P$
   a. $\sup\{a, b\}$ exists in P
   b. $\inf\{a, b\}$ exists in P
   c. Both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in P
   d. none of these

6. An Abelian group G is said to be free abelian group if there is a set $X \subset G$ such that
   a. X is linearly independent.
   b. X generates G
   c. X is both linearly independent and a generating set for G.
   d. None of these

7. $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is a free abelian group having its basis as
   a. $\{(1,0,0), (0,2,0), (0,0,3)\}$
   b. $\{(1,0,0), (2,1,0), (0,0,1)\}$
   c. $\{(2,0,0), (0,1,0), (0,0,1)\}$
   d. $\{(4,0,0), (0,3,0), (0,0,2)\}$

8. Let $X = \{x, y\}$. The word x^{-1} is adjacent to

a. $x^{-1}xyy^{-1}$
b. $x^{-1}xx^{-1}$
c. $yyyxy^{-1}$
d. 1

9. Let $X = \{x, y\}$. The word $xy^{-1}x$ is equivalent to

a. $xxy^{-1}y^{-1}x$
b. $xxx^{-1}y^{-1}yy^{-1}x$
c. $xyy^{-1}xx^{-1}x$
d. $xyxyxy^{-1}x$

10. Let $X = \{x, y\}$. The set of all words in X forms

a. a group
b. an Abelian group
c. a semi group
d. None of these.

11. Let A be any additive abelian group. Then A is

a. a left R- module, $\mathbb{R}$ being ring of real numbers.
b. a left $\mathbb{Q}$- module, $\mathbb{Q}$ being ring of rational numbers.
c. a left $\mathbb{Z}$- module, $\mathbb{Z}$ being ring of integers.
d. None of these.

12. Let R be a subring of a ring S, then

a. R is a left (right) S- module.
b. S is a left (right) R-module.
c. Zero subring (0) is not a left (right) S-module.
d. None of these

13.
If M is an irreducible unital R-module then for any $x \in M$

a. $Rx \subset M$
b. $M \subset Rx$
c. $Rx = M$
d. None of these

14.
For any ring R a left R- module M will be Noetherian if

a. M has acc
b. M has dcc
c. M has both acc and dcc
d. None of these

15. A lattice L is said to be complete if every non empty subset of L has

a. greatest element
b. least element
c. both greatest and least element
d. both supremum and infimum exist in L

16. A lattice L is modular if for all $a, b, c \in L$ $a \geq b$ implies

a. $a \wedge (b \vee c) = b \vee (a \vee c)$
b. $a \wedge (b \vee c) = b \wedge (a \wedge c)$
c. $a \wedge (b \wedge c) = b \wedge (a \vee c)$
d. $a \wedge (b \wedge c) = b \vee (a \wedge c)$

17. If M is an R- module then the set $RM = \{\sum r_i m_i : r_i \in R,\ m_i \in M\}$ is

a. an M-module
b. a submodule of M
c. a submodule of R
d. None of these

18. The submodules of a quotient module M/N are of the form U/N where U is a submodule of

a. M b. N

c. M containing N d. N containing M.

19. If e be an idempotent element of a ring R and R_e a left ideal of R then endomorphism ring $\mathbf{End}_R(R_e)$ is

a. anti-isomorphic to eRe b. isomorphic to eRe

c. homomorphic to eRe d. anti-homomorphic to eRe

20. For a module M if every non empty set S of submodules of M has a maximal element, then

a. M is Artinian b. M is Noetherian

c. M is both Artinian and Noetherian d. None of these

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PART-B : Descriptive

Time: 2 HRS 40 MINS Marks : 50

[Answer question no.(1) & any four (4) from the rest]

1. a. If $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of groups, define direct product $\prod_{\lambda\in\Lambda} G_\lambda$, where 1+3+4+2=10
   (i) $\lambda = \{1, 2, 3, \cdots, n\}$
   (ii) λ is an arbitrary infinite set of indices.

   b. Show that the group $\mathbb{Z} \times \mathbb{Z}$ is free Abelian, where $\mathbb{Z}$ is additive group of integers.

   c. Define a free group on a non-empty set X.

2. a. For any poset (X, ρ) define the converse relation $\bar{\rho}$ on X and hence show that $(X, \bar{\rho})$ is also a poset. 1+3+4+2=10

   b. State and prove the duality principle for any poset (X, ρ).

   c. Draw a lattice diagram for the set L of all factors of 12 with divisibility as the partial order relation and suitable definition of $\vee$ and $\wedge$ for $a, b \in L$.

3. a. When are two posets said to be dually isomorphic? Cite an example of two dually isomorphic posets with justification in support of your answer. 1+3+4+2=10

   b. If G is a cyclic group of prime order then show that the lattice of all subgroups of G is a chain.

   c. For any lattice L and $a, b, c \in L$, show that
   $$a \wedge (a \vee b) = a \quad \text{and} \quad a \vee (a \wedge b) = a$$

4. a. In any lattice L and for $a, b, c \in L$, show that 4+2+4=10
   $$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c).$$
   b. Draw the diagram of the lattice of all factors of 20 under divisibility and show that it is the same as that of the product of two chains with three and two elements.

5. a. Show that $(P(X), \subseteq)$ is a distributive lattice where X is any non-empty set and $\subseteq$ has its usual meaning. 4+6=10

b. Let the group G have subgroups G_i for $1 \leq i \leq n$ such that
(i) G_i commutes with G_j elementwise for $1 \leq i < j \leq n$.
(ii) $G = G_1 G_2 \cdots G_n$
(iii) $G_i \cap G_1 G_2 \cdots G_i G_{i+1} \cdots G_n = \{1\}$ for $1 \leq i \leq n$.
Then $G \cong G_1 \times G_2 \times \cdots \times G_n$.

6. a. If G is a non-zero free Abelian group with a basis of r elements then G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for r factors. 5+1+4=10

b. What do you mean by the statement that "two words u and v in $X \cup X^{-1}$ are equivalent where X is any nonempty set". Prove that the equivalence of words in the set of all words in $X \cup X^{-1}$ is an equivalence relation.

7. a. Prove that every free group is torsion free. 5+4+1=10
b. If A be any additive Abelian group and $\mathbb{Z}$ the ring of all integers, show that A is a left $\mathbb{Z}$ module. Is A a right $\mathbb{Z}$ module also?

8. a. If M is an R-module generated by a set $\{x_1, x_2, \cdots, x_n\}$ and $1 \in \mathbb{R}$ then $M = \sum_{i=1}^{n} Rx_i$. 5+5=10
b. Every homomorphic image M' of a Noetherian module M is Noetherian.

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