REV-01
MSM/58/63

2022/07

M.Sc. MATHEMATICS
SECOND SEMESTER
TOPOLOGY
MSM-201

Duration : 3 hrs. **Full Marks: 70**

(PART-A: Objective)

Time : 20 min. **Marks : 20**

Choose the correct answer from the following: *1X20=20*

1. Let $(X, \mathcal{T}_d)$ be a discrete topological space and $A \subseteq X$. Then
   a. $A \subset \bar{A}$
   b. $\bar{A} \subset A$
   c. $A = \bar{A}$
   d. None of these

2. Let $\mathbf{N} = \{1, 2, 3, 4, \cdots\}$ and $\#(\mathbf{N})$ denote the cardinality of $\mathbf{N}$. Then
   a. $\#(\mathbb{N} \times \mathbb{N}) > \#(\mathbb{N})$
   b. $\#(\mathbb{N} \times \mathbb{N}) < \#(\mathbb{N})$
   c. $\#(\mathbb{N} \times \mathbb{N}) = \#(\mathbb{N})$
   d. None of these

3. Which of the following statements is/ are False?
   a. The finite product of connected space is connected.
   b. Discrete topological space is not a connected space.
   c. The continuous image of a connected space is connected.
   d. Indiscrete topological space is not a connected space.

4. Let $(X, \mathcal{T})$ be a topological space. Then X is a T_3-space if
   a. X is a regular space
   b. X is a normal space
   c. X is a regular and T_1-space
   d. X is a normal and T_1-space

5. Let X_i $(i = 1, 2, \cdots, n)$ be a family of connected spaces. Then the product $\prod_{i=1}^{n} X_i$ is
   a. Connected
   b. Not connected
   c. Connected if $\bigcap_{i=1}^{n} X_i = \phi$
   d. Connected if $\bigcap_{i=1}^{n} X_i \neq \phi$

6. In a T_1 topological space X
   a. No singleton set is closed
   b. Every singleton set is closed
   c. Each singleton set is both closed and open
   d. None of these

7. Let X be a discrete topological space. Which of the following sets is/are dense on X?
   a. X
   b. A non-empty subset of X
   c. ϕ
   d. All the above

8. Let X be a discrete topological space and $A \subset X$. Then ∂A is equal to

a. Subset of X
b. Proper subset of X
c. X
d. Empty set

9. Discrete topological space is

a. Compact.
b. Not compact.
c. Not Connected.
d. Connected.

10. Let $Y = [0,1] \cup (2,3)$ be a subspace topology of $\mathbb{R}$. Then

a. $[0,1]$ is closed in Y.
b. $(2,3)$ is closed in Y.
c. Both $[0,1]$ and $(2,3)$ are closed in Y.
d. Neither $[0,1]$ nor $(2,3)$ is closed in Y.

11. Let X be a non-empty set and $\mathcal{T}_1, \mathcal{T}_2$ be topological spaces on X. Consider the following two statements

P: $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology on X.

Q: $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X

a. P false Q true
b. P true Q false
c. P and Q both true
d. P and Q both false

12. "The unit open interval $]0,1[$ is equivalent to the set $\mathbb{R}$ of all real numbers". The statement is

a. False
b. True
c. meaningless
d. none of these

13. Let $(X, \mathcal{T})$ be a co-finite topological space. Then

a. X is a compact space.
b. X is not a compact space.
c. X is a compact space if X is a finite set.
d. X is a compact space if X is an infinite set.

14. Let X be a connected topological space and Y be a topological space. If $f : X \to Y$ is a homeomorphism. Then

a. $f(X) = Y$ is connected
b. $f(X) \neq Y$, $f(X)$ is connected
c. $f(X) = Y$ is not connected
d. $f(X) \neq Y$, $f(X)$ is not connected

15. Every metric space X is

a. a T_1 space
b. a T_2 space
c. Both a T_1 and a T_2 space
d. Neither a T_1 space nor a T_2 space

16. Consider the following statements:
P: The usual topology on $\mathbb{R}$ is not locally compact.
Q: Any discrete topological space is locally compact.

a. P False, Q True
b. P True, Q False
c. Both are True
d. Both are False

17. A topological space $(X, \mathcal{T})$ is second countable if

a. There is a countable number of open sets in X.
b. There is a base for $\mathcal{T}$ having a countable number of open sets.
c. There is a countable local base at every point of X.
d. None of these

18. Which of the following is/are true?

a. $[-1,1] \times [0,1]$ is compact but not locally compact.
b. $[-1,1] \times [0,1]$ is locally compact but not compact.
c. $[-1,1] \times [0,1]$ is both compact and locally compact.
d. $[-1,1] \times [0,1]$ is neither compact nor locally compact.

19. Let $A = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 1\}$ and $B = \{(2,2)\}$. Let $f: \mathbb{R} \times \mathbb{R} \to [0,1]$ as

$$f(x) = \begin{cases} 1, & \forall x \in A \\ 0, & \forall x \in B \end{cases}$$

Then f is

a. Continuous function
b. not a continuous function
c. Information is insufficient
d. None of these

20. The topological space $\mathbb{R}$ with discrete topology $\mathcal{D}$ is

a. First countable
b. Second countable
c. Both first and second countable
d. Neither first countable nor second countable.

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PART-B : Descriptive

Time: 2 HRS 40 MINS Marks : 50

[Answer question no.(1) & any four (4) from the rest]

1. Prove or disprove the following statements: 2×5=10
   (a) The set A of all algebraic numbers is denumerable. (A real number r is defined to be an algebraic number if r satisfies a polynomial equation of the form $a_0 + a_1x + a_2x^2 + \cdots + a_mx^m = 0$, where $a_0, a_1, \cdots, a_m$ are integers.)
   (b) Co-finite topology on a finite set X is the same as the discrete topology on it.
   (c) The set $\mathbb{R}$ of all real numbers with usual topology $\mathcal{U}$ is second countable.
   (d) Every T_2-space is also a T_1-space.
   (e) The topological space $\mathbb{R}$ with usual topology $\mathcal{U}$ is separable.

2. a. Let A, B be the subspace topologies of the topological spaces X and Y respectiviely. Prove that the product topology and subspace topology on $A \times B$ are same. Is the result true for ordered topology? 3+1+4+2=10

   b. Let $\mathbb{R}$ denote the usual topology and $\mathbb{R}_\ell$ denotes the lower limit topology on the real line $\mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}_\ell$ be defined as $f(x) = x$, $\forall x \in \mathbb{R}$. Prove or disprove that f is a homeomorphism.

   c. Let $Y = (0, 1]$ be a subspace of the usual topology on $\mathbb{R}$. Find the closure of the set $A = \left(0, \frac{1}{2}\right)$ in Y.

3. a. Let A be any subset of a second countable space X. If $\mathcal{G}$ is an open cover of A then prove that $\mathcal{G}$ is reducible to a countable cover. 5+5=10

   b. Prove that – A separable metric space is second countable.

4. Let $(X, \mathcal{T})$ be a topological space. Define a relation R on X as follows: $R = \{(x, y) \in X \times X : x, y \in E_{xy}\}$, where E_{xy} is a connected subset of X. 4+6=10
   (i) Prove that – R is an equivalence relation.

   b. Let $(X, \mathcal{T})$ be a topological space and $\alpha \notin X$. Define $X^+ = X \cup \{\alpha\}$ and $\mathcal{T}^+ = \{G \subset X^+ : \alpha \in G \;\&\; X^+ - G$ is closed and compact in $X\}$. Show that $(X^+, \mathcal{T}^+)$ is a topological space.

5. a. Prove that a topological space X is normal if and only if for every closed set F and every open set H containing F there exists an open set G such that $F \subset G \subset \bar{G} \subset H$. 5+5=10

b. Let $\mathcal{D}$ denote the set of all dyadic fraction in $[0,1]$. Prove that $\bar{\mathcal{D}} = [0,1]$.

6. Check the connectedness of the following topological space (Explain): 2×5=10
(i) Discrete topology on X.
(ii) Finite- complement topology on an infinite set X.
(iii) The real line with lower limit topology.
(iv) A topology $\mathcal{T} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}\}$ on $X = \{a, b, c, d\}$.
(v) A topology $\mathcal{T} = \{\phi, X, \{a\}, \{b, c\}\}$ on $X = \{a, b, c\}$.

7. **a.** Consider the topological space $(X, \mathcal{T})$, where $X = \{a, b, c, d, e\}$ and $\mathcal{T} = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{a, b\}$. Find $\bar{A}$ the closure of A in X. Is A dense in X? Is X a separable topological space? 3+1+1+3+1+1=10

b. Find the topology $\mathcal{T}$ on $X = \{a, b, c, d\}$ generated by the class C of subsets of X given by $C = \{\{a\}, \{b\}\}$. Mention the subbase and base for the topology $\mathcal{T}$ on X.

8. **a.** Prove that - The continuous image of a compact space is compact. 4+4+2=10
b. Prove that - A closed subspace of a compact space is compact.
c. Construct an open cover on $\mathbb{R}^2$ with usual topology.

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