6. **a.** Show that the transformation $w = \frac{z+i}{z-i}$ transform $|w| \le 1$ into the lower half plane $Im(z) \le 0$. 5+5 =10

b. Find the bilinear transformation which maps the points $z_1 = 1, z_2 = 0,\ z_3 = -1$ into the points $w_1 = i, w_2 = \infty, w_3 = 1$.

7. **a.** Prove that $\lim_{z \to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i} = 4 + 4i$ 5+5=10

b. Show that the points z_1, z_2 are the inverse points with respect to the circle $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$ if $z_1\overline{z_2} + b\overline{z_2} + \bar{b}z_1 + c = 0$

8. **a.** Define entire function. Give an example. 2+8=10

b. State and prove Jensen's formula.

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REV-00 2018/06
MSM/39/44

M. Sc. MATHEMATICS
SECOND SEMESTER
COMPLEX ANALYSIS
MSM - 203

(Use Separate Answer Scripts for Objective & Descriptive)

Duration : 3 hrs. **Full Marks : 70**

(PART-A : Objective)

Time : 20 min. **Marks : 20**

Choose the correct answer from the following: ***1×20=20***

1. The bilinear transformation which maps the point $z_1 = 2, z_2 = i, z_3 = -2$ and $w_1 = 1$, $w_2 = i,\ w_3 = -1$ is
   a. $w = \dfrac{3z+2i}{iz-6}$ b. $w = \dfrac{3z-2i}{iz+6}$
   c. $w = \dfrac{3z+2i}{iz+6}$ d. $w = \dfrac{3z-2i}{iz-6}$

2. Suppose $f(z) = u(x,y) + iv(x,y)$ is an analytic function in a domain D. If u and v are harmonic in D, then which of the following is true?
   a. $u_{xy} + u_{yx} = 0, v_{xy} + v_{yx} = 0.$ b. $u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$
   c. $u_{xy} + v_{yx} = 0, v_{xy} + u_{yx} = 0$ d. $u_{xx} + v_{yy} = 0, u_{yy} + v_{xx} = 0$

3. Polar form of Cauchy-Riemann is:
   a. $v_\theta = ru_r, u_\theta = -rv_r$ b. $v_r = ru_\theta, u_\theta = -rv_r$
   c. $v_\theta = ru_r, u_r = -rv_\theta$ d. $v_\theta = -ru_r, u_\theta = rv_r$

4. Consider the following statements:
   P: The derivative of $f(z) = |z|^2$ exist for all z except $z = 0$.
   Q: The derivative of $f(z) = |z|^2$ exist only at $z = 0$.
   a. Only P is true b. Only Q is true
   c. Both P and Q are true. d. Both P and Q are false.

5. The linear fractional transformations that maps $Im(z) \ge 0$ onto $|w| \le 1$ is of the form:
   a. $w = e^{i\alpha}\dfrac{z-z_0}{z-\bar{z}_0}, Im(z_0) \ge 0$ b. $w = e^{i\alpha}\dfrac{z-z_0}{z+\bar{z}_0}, Im(z_0) \ge 0$
   c. $w = e^{i\alpha}\dfrac{z+z_0}{z-\bar{z}_0}, Im(z_0) \ge 0$ d. None of these

6. Let $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half plane and $D = \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disc. Suppose that f is a bilinear transformation; which maps H conformally onto D. Suppose that $f(2i) = 0$. Consider the following statement:
   P: f has a simple pole at $z = -2i$.
   Q: $|f(2+2i)| = \frac{1}{\sqrt{5}}$.
   a. Both P and Q are false. b. P is true but Q is false.
   c. P is false but Q is true. d. Both P and Q are true.

7. The harmonic conjugate of $u(x,y) = x^3 - 3xy^2$is:
   a. $3xy^2 - y^3 + c$ b. $3xy^2 + y^3 + c$
   c. $3xy^2 + y^3 + c$ d. $3x^2y - y^3 + c$

8. The radius of convergence of the series $\sum_0^\infty z^n$ is
a. 0 b. 1 c. 2 d. 3

9. The curve represented by $\frac{z-z_1}{z-z_2} = c$, where $c \neq 1$ is
a. Straight line b. Circle c. Ellipse d. None of these

10. The number of distinct cross ratios is:
a. 4! b. 6 c. 4 d. None of these

11. The value of $\oint \frac{dz}{z-2}$ over a simple closed curve C, where C is $|z| = 3$.
a. 0 $b.\ 2\pi i$ $c.\ 4\pi i$ $d.\ 8\pi i$

12. The value of $\frac{1}{2\pi i}\oint \frac{e^z}{z-2}$ over C, where C is the circle $|z| = 3$ is:
a. 1 $b.\ e$ $c.\ e^2$ d. None of these

13. The condition that the four points z_1, z_2, z_3, z_4 are concyclic is
a. $\frac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ is purely real
b. $\frac{(z_4-z_3)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ is purely real
c. $\frac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ is purely imaginary
d. None of these

14. The value of $\oint \frac{dz}{z-a}$ over a simple closed curve C, where a is outside C
a. 0 b. 3 c. 2 d. None of these

15. A finite order of an entire function $f(z)$ satisfies the inequality $log M(r) < r^k$, where k is the positive integer and $|z| = r$. Here $M(r)$ defines as the
a. Maximum modules
b. Minimum modules
c. Either maximum or minimum modules
d. None of these

16. If an entire function has no singularity at infinity, then by Liouville's theorem, the function is a/an
a. Increasing function
b. Decreasing function
c. Constant function
d. None of these

17. If $f(z) = \frac{\phi(z)}{\theta(z)}$, then $f(z)$ has poles at
a. Poles of $\theta(z)$ $b.$ Zeros of $\theta(z)$ $c.$ Zeros of $\emptyset(z)$ $d.$ None of these

18. Residue of $f(z)$ at a simple pole $z = a$ is
a. $\lim_{z\to a} zf(z)$
b. $\lim_{z\to a}(z-a)f(z)$
c. $\lim_{z\to a}(z-a)^2 f(z)$
d. None of these

19. If $f(z) = \frac{1}{1+z^2}$, then residue at $z = i$ is
a. $1/2$ $b.\ 1/2i$ $c.\ \frac{i}{2}$ $d.$ None of these

20. If $f(z)$ is an analytic function within and on a simple closed curve C, the point giving the maximum of $|f(z)|$ can be
a. Within C
b. Outside C
c. On the boundary C and not within it
d. On the boundary and within C

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(PART-B : Descriptive)

Time : 2 hrs. 40 min. **Marks : 50**

[Answer question no.1 & any four (4) from the rest]

1. State and prove Taylor's theorem. Describe the region of convergence for the Taylor series. 8+2=10

2. a. Find the Laurent's series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of the convergence of the series. 6+4=10
(i) $\frac{e^{2z}}{(z-1)^3}, z = 1$ (ii) $(z-3)sin\frac{1}{z+2}, z = -2$ (iii) $\frac{z-sinz}{z^3}, z = 0$
b. State the Rouche's theorem. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circle $|z| = 1$ and $|z| = 2$.

3. a. Define analytic function with example. State necessary and sufficient condition for $f(z)$ to be analytic. 4+6 =10
b. Show that $u(x,y) = y^3 - 3x^2y$ is harmonic in some domain and find the harmonic conjugate of $u(x,y)$. Also, find the corresponding analytic function $f(z)$ in terms of z.

4. a. State Picard's little theorem and Bloch's theorem. Define Bloch's constant. 3+5 +2=10
b. Find the fixed points and normal form of the transformation $f(z) = \frac{z-1}{z+1}$. Classify the nature of the transformation.
c. Define Isogonal transformation. Give an example of isogonal transformation.

5. a. If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C, then prove that 7+3=10
$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$ over C.
Here C is traversed in positive direction
b. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)}$