8. a. Determine which of the polynomials below are irreducible over $\mathbb{Q}$? (2x3)+4=10

(i) $x^2 + 9x^4 + 12x^2 + 6$

(ii) $x^4 + x + 1$

(iii) $x^5 + 5x^2 + 1$

b. Prove that - $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$.

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M.Sc. MATHEMATICS
FIRST SEMESTER
ABSTRACT ALGEBRA-I
MSM-103

(Use separate answer scripts for Objective & Descriptive)

Duration: 3 hrs. **Full Marks: 70**

[PART-A: Objective]

Time: 20 min. **Marks: 20**

Choose the correct answer from the following: *1x20=20*

1. The set $\{1, 2, \cdots, n-1\}$ is a group under multiplication modulo n iff n is:
   - a. Composite
   - b. Prime
   - c. For any integer
   - d. None of these
2. The identity element of $GL(2, \mathbb{R})$ under matrix multiplication is:
   - a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   - b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
   - c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
   - d. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3. If $G = Z(G)$ then:
   - a. $Z(G)$ is a subgroup of G
   - b. G is an Abelian group
   - c. G is a group but may not be Abelian
   - d. None of these
4. The value of $R_{270}R_0$ and $R_{270}D$ are:
   - a. R_{270}& V
   - b. R_{270}& H
   - c. R_{270}&R_{180}
   - d. R_{270}&R_{90}
5. Let G be a group and a be an element of G such that $a^{12} = e$, then $\text{ord}(a)$ is:
   - a. 12
   - b. Divisor of 12
   - c. ≤ 12
   - d. None of these
6. A generator of $\mathbb{Z}_{12}$ is:
   - a. 3
   - b. 6
   - c. 5
   - d. 10
7. Consider the following statements:

   P : The permutation $(12)(134)(152)$ is an odd permutation.

   Q : The symmetric group S_7 contain an element of order 14.
   - a. P is true but Q is false
   - b. P is false and Q is true
   - c. P and Q both are true
   - d. P and Q both are false
8. The group $\mathbb{Z}_8^*$ is:
   - a. Cyclic and all of its subgroups are also cyclic.
   - b. Non-cyclic but all of its subgroups are cyclic.
   - c. Cyclic and some of its subgroups are cyclic.
   - d. Non-cyclic but some of its subgroups are cyclic.
9. Let G and $\bar{G}$ be a group and $\phi: G \to \bar{G}$ be a group isomorphism, then

   P : If H is a subgroup of G then $f(H) = \{\phi(h): h \in H\}$ is a subgroup of G.

   Q : If G is cyclic then $\bar{G}$ is also cyclic. But converse is not true.
   - a. P is true but Q is false
   - b. P is false and Q is true
   - c. P and Q both are true
   - d. P and Q both are false

10. Let H be a subgroup of a group G and $a, b \in G$. Then:

a. $aH = bH$ iff $ab \in H$ b. $aH = bH$ iff $ab^{-1} \in H$

c. $aH = bH$ iff $a^{-1}b \in H$ d. $aH = bH$ iff $ba \in H$

11. Consider the following statements:

P : A group of prime order is cyclic.

Q : A subgroup H of a group G is normal in G iff $xHx^{-1} \subseteq H, \forall x \in G$.

a. P is true, Q is false b. Q is true, P is false

c. P and Q both are false d. P and Q both are true

12. Consider the following statements:

P : If $\frac{G}{Z(G)}$ is cyclic then G is cyclic.

Q : if G is finite Abelian group and p is a prime number such that $p \mid |G|$ then G has an element of order p.

a. P is true, Q is false b. P is false, Q is true

c. P and Q both are true d. P and Q both are false

13. Consider the following statements:

P : $n\mathbb{Z}$ is prime ideal of $\mathbb{Z}$ iff n is prime.

Q : $< x^2 + 1 >$ is a prime ideal in $\mathbb{Z}_2[x]$.

a. P is true, Q is false b. P is false, Q is true

c. P and Q both are true d. P and Q both are false

14. Up to isomorphism, the number of Abelian groups of order 108 is:

a. 12 b. 9 c. 6 d. 5

15. In the group of all invertible 4×4 matrices with entries in the field of 3 elements, any 3-Sylow subgroup has cardinality:

a. 3 b. 81 c. 243 d. 729

16. Consider the polynomial function $f(x) = x^4 + 1$. Which of the following is true?

a. $f(x)$ is irreducible over $\mathbb{Q}$ b. $f(x)$ is irreducible over $\mathbb{Z}_2$

c. $f(x)$ is irreducible over $\mathbb{Z}_3$ d. $f(x)$ is irreducible over $\mathbb{Z}_5$

17. Consider the following statements:

P : Every Euclidean domain is a principal ideal domain.

Q : Every Euclidean domain is a unique factorization domain.

a. P is true, Q is false b. P is false, Q is true

c. Both P and Q are true d. None of these

18. The number of elements in the field $\frac{\mathbb{Z}_2[x]}{<x^3+x+1>}$ is:

a. 9 b. 8 c. 6 d. None of these

19. Let G be a non-Abelian group. Then, its order can be:

a. 25 b. 55 c. 9 d. 35

20. Which of the following is class equation of a group of order 10?

a. 1+1+1+2+5=10 b. 1+2+3+4=10

c. 1+2+2+5=10 d. 1+1+2+2+2+2=10

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(PART-B : Descriptive)

Time : 2 hrs. 40 min. **Marks : 50**

[Answer question no.1 & any four (4) from the rest]

1. Define Abelian group. Give an example of a non-Abelian group and explain. 4+6=10

2. a. Let G be an Abelian group with identity e and let n be some integer. Prove that the set of all elements of G that satisfy the equation $x^n = e$ is a subgroup of G. 2+3+3+2=10

b. Let $\alpha = (12)(345)$ and $\beta = (1\ 23456)$. Find the value of:

(i) α^{-1} and β^{-1}

(ii) $\alpha\beta$ and $\beta\alpha$

c. Let $\alpha \in S_7$ and suppose $\alpha^4 = (2143567)$. Find α.

d. Find the number of element of order 2 in S_5.

3. a. Determine the number of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$. 5+5=10

b. Suppose that G is a non-Abelian group of order p^3 and $Z(G) \neq \{e\}$. Prove that $|Z(G)| = p$.

4. a. State the first Isomorphism theorem. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and Ker $\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9. 1+5+4=10

b. Determine which of the following cannot be the class equation of a group.

(i) 10=1+1+1+2+5

(ii) 4=1+1+2

(iii) 8=1+1+3+3

(iv) 6=1+2+3

5. a. Let R be a ring and let $A = \{x \in R : ax = xa, \forall a \in R\}$. Prove that A is a subring of R. 5+2+3=10

b. Define Integral domain and field. Prove that - A finite integral domain is a field.

6. a. Define prime and maximal ideal. 2+5+3=10

b. Define Euclidean domain and Principal ideal domain. Prove that - Every Euclidean domain is principal ideal domain.

c. Prove that - the ideal $< x^2 + 1 >$ is maximal ideal in $\mathbb{R}[x]$.

7. a. Prove that - A group of order 99 is Abelian. 4+2+4=10

b. Find the order of Sylow-2 subgroup of S_6.

c. Let G be a group of order 45. Prove that -

(i) G has a normal subgroup of order 5.

(ii) G has a normal subgroup of order 9.

PTO